TWO-DIMENSIONAL SPREADING OF A CLOUD OF CONDUCTING GAS IN A MAGNETIC FIELD SELF-SIMILAR SOLUTION

V. I. Yakovlev

UDC 538.4

A self-similar solution of the problem on the spreading in a magnetic field of a cloud of conducting gas, having the shape of a cylinder of noncircular cross section, is constructed. The cylindrical surface of the gas is restrained by a nonconducting sheath that spreads according to a prescribed law. The shape of the transverse cross section of the cylindrical cloud is determined from the solution. Cross sections obtained for a concrete case are represented in graphic form.

The character of the flows in many magnetohydrodynamic devices makes it necessary to investigate two-dimensional nonsteady motions in magnetogasdynamics. A great many papers are devoted to theoretical discussions of similar problems. Under the assumption that the magnetic Reynolds number R_m and the magnetohydrodynamic interaction parameter are small, studies have been made of the eddy currents generated in the gas behind the shock wave in the region of inhomogeneity of the external magnetic field and of their influence on the propagation speed of the shock wave [1] and on the braking of a plasma cloud in a channel [2]. For $R_m = \infty$ the behavior of a two-dimensional Z-pinch with the formation of a plasma focus has been studied in [3] and the parameters of an axial plasma jet were obtained.

In two-dimensional, as in one-dimensional, problems, it is important to consider finite values of the number R_m, since in the first place this number is by no means small in certain experimental apparatus [4], and secondly, as has been shown in [5], even for flows having an R_m that is initially small, the development of small but finite perturbations in the conductivity can lead to a substantial rearrangement of the flow with the formation of high-temperature layers, and this results in an increase in the effective value of the magnetic Reynolds number, while the induced magnetic fields become sizeable.

In the general case such problems can obviously be investigated only by the application of numerical methods, but at the same time the need for exact solutions remains.

One of the best-known methods of obtaining exact solutions is the method of self-similar solutions. Usually one considers problems in whose original formulation all unknown functions depend on only two independent variables, and the introduction of a self-similar variable reduces the problem to the integration of ordinary differential equations [6, 7].

In the case of nonsteady two-dimensional problems of magnetogasdynamics with three independent variables the possibility of obtaining an exact solution of the self-similar problem is not evident; therefore the construction of such a solution is of interest.

In the present paper, we give an example of a self-similar solution of the problem for the case of two-dimensional spreading of a cloud of conducting gas, contained in a nonconducting, mobile sheath.

Suppose that we have a two-dimensional cloud of conducting gas in the shape of a cylinder with generators parallel to the z axis, whose lateral surface is restrained by a nonconducting sheath. The cross section of the cylinder in a plane $z = \text{const}$ is some closed curve whose shape is to be found from the solution. The entire system is situated in an external magnetic field that has only a z-component.

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 29-36, March-April, 1973. Original article submitted August 4, 1972.

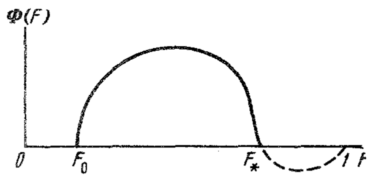

Fig. 1

At some instant let the sheath start to expand, remaining geometrically similar in the shape of its cross section, while the strength of the external magnetic field begins to diminish. The displacement of the sheath results in motion of the gas, in which closed electric currents, lying in the planes z = const, are induced. The magnetic field of the currents will be combined with the external magnetic field and will change it inside the cloud. The conduction currents will have no effect on the field outside the cloud. The influence of displacement currents on the magnetic field outside the cloud is also negligibly small (since the velocity with which the cloud spreads is small in comparison with the speed of light).

After a certain time the characteristic dimension d of the cloud becomes substantially larger than the initial dimension d_0 while the strength H of the external field becomes smaller than H_0. At this time the solution ceases to depend on D_0, H_0, and on the initial conditions in the cloud (with the exception of a small neighborhood of the origin). The solution that is obtained after the self-similar regime has become established will be constructed in this paper.

Using a cylindrical coordinate system (r, φ, z), we have $\partial/\partial z \equiv 0$ for the case under consideration. Let r_* be the r-coordinate of a boundary point of the cloud; according to our stipulation $r_* = r_*(\varphi, t)$.

We seek a solution with the velocity field

$$\mathbf{v} = v(r,t)\,\mathbf{e}_r, \qquad v(r,t) = rf'(t)/f(t) \tag{1}$$

and with a magentic field $\mathbf{H} = H(r, \varphi, t)\mathbf{e}_z$.

A motion with the velocity field (1) ensures the geometrical similarity of the boundary of the cloud (which is necessary for a self-similar process) at any time t.

The equations of magnetogasdynamics with (1) taken into account have the form

$$\begin{gathered}\frac{d\rho}{dt} + \frac{\rho}{r}\frac{\partial}{\partial r}(rv) = 0\\ \rho\frac{dv}{dt} = -\frac{\partial}{\partial r}\left(P + \frac{H^2}{8\pi}\right), \qquad \frac{\partial}{\partial \varphi}\left(P + \frac{H^2}{8\pi}\right) = 0\\ \frac{\partial H}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}(rvH) + \frac{1}{r}\frac{\partial}{\partial r}\left(\nu_m r\frac{\partial H}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \varphi}\left(\frac{\nu_m}{r}\frac{\partial H}{\partial \varphi}\right)\\ \frac{d}{dt}\left(\frac{P}{\rho^\gamma}\right) = 0 \qquad \left(\nu_m = \frac{c^2}{4\pi\sigma}, \quad \sigma = \Omega P^n \rho^m\right)\end{gathered} \tag{2}$$

where ν_m is the magnetic viscosity, σ is the conductivity, and c is the speed of light.

Equations (2) are written under the assumption that the medium is a perfect gas with constant specific heats, that viscosity and heat conductivity are absent, and that the electrical conductivity is related to the pressure P and the density ρ through a power-law dependence. In the energy equation Joule heat is not taken into consideration.

Taking the transverse dimension of the cloud to be equal to zero at the instant t = 0, we shall assume that the law by which the sheath is displaced is given in the form

$$r_* = lt^\beta \psi(\varphi) \tag{3}$$

while the strength of the external magnetic field varies according to the law

$$H_*(t) = \Lambda t^\alpha \tag{4}$$

Here $\psi(\varphi)$ is a dimensionless periodic function of the angle φ, l and Λ to be determined, $[l] = \mathrm{cm}\cdot\mathrm{sec}^{-\beta}$, and $[\Lambda] = \mathrm{g}^{1/2}\cdot\mathrm{cm}^{-1/2}\cdot\mathrm{sec}^{-(1+\alpha)}$, α and β are dimensionless constants. Moreover, let M_0 be the mass gas occupying a unit length of the cylindrical cloud, $[M_0] = \mathrm{g}\cdot\mathrm{cm}^{-1}$. Thus, according to [6], the determining parameters of the problem are

$$r,\ \varphi,\ t,\ \Lambda,\ 4\pi\Omega/c^2,\ l,\ M_0 \qquad ([4\pi\Omega/c^2] = \mathrm{g}^{-(n+m)}\,\mathrm{cm}^{n+3m-2}\cdot\mathrm{sec}^{2n+1})$$

For $\alpha = -1$, $\beta = -(2n+1)/2(m-1)$ the problem is self-similar. At the same time, from the parameters r, t, Λ, $4\pi\Omega/c^2$, l, M_0, one can form the single dimensionless quantity

$$\xi = rl^{-1}t^{(2n+1)/2(m-1)} \tag{5}$$

which, according to (3), assumes the following value at the outer boundary of the cloud

$$\xi_* = \psi(\varphi) \tag{6}$$

The desired solution will have the form

$$v(r,t) = -\frac{2n+1}{2(m-1)}\frac{r}{t} = -\frac{2n+1}{2(m-1)} l t^{-(2n+1)/2(m-1)-1}\xi$$
$$H(r,\varphi,t) = \sqrt{8\pi}\Lambda t^{-1} h(\xi,\varphi), \quad P(r,\varphi,t) = \Lambda^2 t^{-2} p(\xi,\varphi) \tag{7}$$
$$\rho(r,t) = \frac{M_0}{r^2}\rho_1(\xi) = \frac{M_0}{l^2} t^{(2n+1)/(m-1)}\rho_0(\xi)$$

The velocity v satisfies the required boundary condition

$$v(r,t)\,|_{r=r_*(\varphi,t)} = \partial r_* / \partial t$$

and in the expression for the density we omit the dependence on φ, which is obvious from Eqs. (2).

Insertion of (7) into (2) determines equations for the functions $\rho_0(\xi)$, h (ξ, φ), p (ξ, φ). At the same time the first of Eqs. (2) is satisfied identically, while the fifth equation of (2) reduces to the relationship

$$\gamma = -2(m-1)/(2n+1) \tag{8}$$

where γ is the adiabatic exponent.

The second, third, and fourth equations of the system (2) reduce to the following:

$$\frac{M_0}{\Lambda^2}\frac{\gamma-1}{\gamma^2}\xi\rho_0(\xi) = \frac{\partial}{\partial\xi}(p+h^2), \quad \frac{\partial}{\partial\varphi}(p+h^2) = 0 \tag{9}$$

$$\frac{1}{\xi}\frac{\partial}{\partial\xi}\left[p^{-n}\rho_0^{-m}\xi\frac{\partial h}{\partial\xi}\right] + \frac{1}{\xi^2}\frac{\partial}{\partial\varphi}\left[p^{-n}\rho_0^{-m}\frac{\partial h}{\partial\varphi}\right] = \frac{2-\gamma}{\gamma}Nh(\xi,\varphi) \tag{10}$$
$$\left(N = \frac{4\pi\Omega}{c^2} l^2\Lambda^{2n}\left(\frac{M_0}{l^2}\right)^m\right)$$

In addition to Eqs. (9), (10), the required solution must satisfy a boundary condition on the magnetic field

$$H(r,\varphi,t)/_{r=r_*(\varphi,t)} = H_*(t)$$

which, from the second of (7), with account taken of (4), reduces to the form

$$h[\xi_*(\varphi),\varphi] = (8\pi)^{-1/2} \tag{11}$$

We seek a particular solution of Eqs. (9), (10) of the form

$$h(\xi,\varphi) = R(\xi)F(\varphi), \quad p(\xi,\varphi) = R^2(\xi)[1-F^2(\varphi)] \tag{12}$$

the functions R(ξ), $\rho_0(\xi)$, F (φ) having to satisfy, according to (9), (10), the equations

$$[R^2(\xi)]' = \frac{M_0}{\Lambda^2}\frac{\gamma-1}{\gamma^2}\xi\rho_0(\xi) \tag{13}$$
$$\frac{1}{\xi R}\frac{d}{d\xi}\left[\frac{\xi R'(\xi)}{R^{2n}\rho_0^{\ m}}\right]\frac{1}{(1-F^2)^n} + \frac{1}{\xi^2 R^{2n}\rho_0^{\ m}}\frac{1}{F(\varphi)}\frac{d}{d\varphi}\left[\frac{F'(\varphi)}{(1-F^2)^n}\right] = \frac{2-\gamma}{\gamma}N$$

the latter of which is equivalent to the following:

$$\xi^2 R^{2n}(\xi)\rho_0^{\ m}(\xi) = a, \quad \frac{1}{\xi R(\xi)}\frac{d}{d\xi}\left[\frac{\xi R'(\xi)}{R^{2n}(\xi)\rho_0^{\ m}(\xi)}\right] = b \tag{14}$$

$$\frac{ab}{(1-F^2)^n} + \frac{1}{F(\varphi)}\frac{d}{d\varphi}\left[\frac{F'(\varphi)}{(1-F^2)^n}\right] = aN\frac{2-\gamma}{\gamma} \tag{15}$$

Here a, b are arbitrary constants, on which the restriction (17) is imposed. Solutions of Eqs. (13), (14) have the form

$$R(\xi) = a^{1/2(m+n)}\left(\frac{M_0}{2\Lambda}\frac{\gamma-1}{\gamma^2}\frac{m+n}{m-1}\right)^{m/2(m+n)}\xi^{(m-1)/(m+n)} \tag{16}$$
$$\rho_0(\xi) = a^{1/(m+n)}\left(\frac{M_0}{2\Lambda}\frac{\gamma-1}{\gamma^2}\frac{m+n}{m-1}\right)^{-n/(m+n)}\xi^{-2(n+1)/(m+n)}$$

These solutions satisfy the second equation of (14) when

$$ab = \frac{(m-1)(3m+2n-1)}{(m+n)^2} \tag{17}$$

It should be noted that solutions (16) were obtained under the assumption that $m \neq 0$, $m+n \neq 0$. The case $m=0$ is rejected because it lacks physical interest; the case $m+n=0$ is not considered for the reason

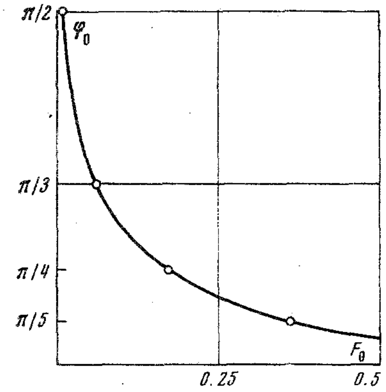

Fig. 2

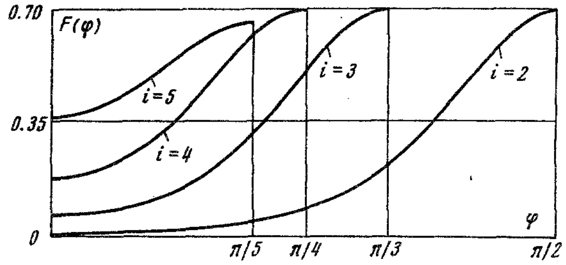

Fig. 3

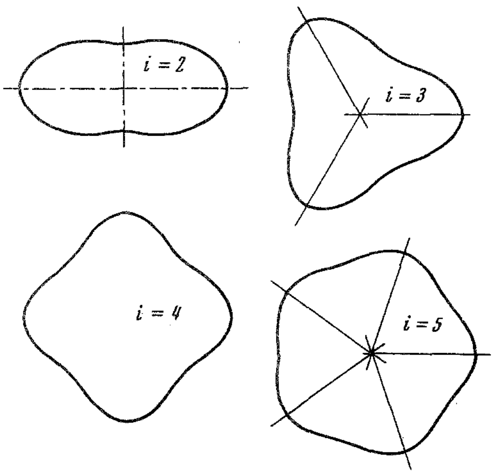

Fig. 4

that in this case compatibility of Eqs. (13), (14) is achieved only for $n = -1$ (i.e., only when the conductivity is inversely proportional to the temperature), which is also without physical interest.

With (12), (16) taken into account, the boundary condition (11) assumes the form

$$\xi_*^{(m-1)/(m+n)}(\varphi)\, F(\varphi) = a^{-1/2(m+n)} \left(\frac{M_0}{2\Lambda}\,\frac{\gamma-1}{\gamma^2}\,\frac{m+n}{m-1}\right)^{-m/2(m+n)} (8\pi)^{-1/2}. \quad (18)$$

This is, in fact, used to find $\xi_*(\varphi)$, i.e., for the determination of the shape of the transverse cross section of the cylindrical cloud, after solution of the equation for $F(\varphi)$.

We now turn to Eq. (15). From the form of solution (12) and the condition (18) it is evident that a solution of this equation has physical meaning if it is periodic and is restricted between the limits $0 \leq F \leq 1$. The period must coincide with one of the values $2\pi/i\,(i = 1, 2, 3, \ldots)$, where, according to (18), i determines the number of "lobes" in the cross section of the cloud. Denoting by $F_0\,(0 < F_0 < 1)$ the minimum value of $F(\varphi)$, we can write the initial conditions for Eq. (15) in the form

$$F(0) = F_0, \qquad F'(0) = 0. \quad (19)$$

The value of F_0 is determined from the condition that the period of the function $F(\varphi)$ should coincide with $2\pi/i$.

Equation (15) contains two constants, ab and aN. The first of these constants is determined by condition (17), and for the values of m, n chosen below (20) it takes the value $ab = 35$. The second constant aN is arbitrary, and, as shown below, it satisfies the condition $(2-\gamma)\gamma^{-1} \cdot aN > ab$.

The constant aN is related to the magnetic Reynolds number. As there are no length and velocity scales in the problem under consideration, the number R_m can be calculated only in terms of the running parameters $(\xi, \varphi, t) = \Omega P^n(\xi, \varphi, t)\, \rho^m(\xi, \varphi, t), v(\xi, t), r(\xi, t)$ of some fixed particle. It is found that for the solution obtained this number does not depend on ξ, t and is determined only by the parameter φ of the particle in question. In fact, insertion of the solutions P, ρ, ν, r into the expression for R_m leads to the following result:

$$R_m = 4\pi\Omega P^n \rho^m v r c^{-2} = aN(1 - F^2)^n.$$

However, it is clear that prescribing aN is to a certain extent equivalent to prescribing the number R_m.

Introducing the new function

$$u(F) = \frac{F'(\varphi)}{(1-F^2)^n}, \qquad \frac{d}{d\varphi} = F'(\varphi)\frac{d}{dF} = u(1-F^2)^n \frac{d}{dF}$$

we can integrate Eq. (15) once, with the result

$$u^2 = \frac{2-\gamma}{\gamma} aNJ_1(F) - abJ_2(F) + D$$

$$\left(J_1(F) = \int \frac{2F\,dF}{(1-F^2)^n}, \quad J_2(F) = \int \frac{2F\,dF}{(1-F^2)^{2n}}\right)$$

where D is an arbitrary integration constant. The integrals $J_1(F)$ and $J_2(F)$ assume different forms, depending on the exponent n.

In choosing n it must be kept in mind that the constants γ, m, n must satisfy condition (8), which, though restricting the arbitrariness of these constants, nevertheless leaves the possibility of considering physically meaningful cases. For example, we can take

$$\gamma = {}^5/_3, \qquad m = -{}^3/_2, \qquad n = 1 \tag{20}$$

With these values of m, n the dependence of the conductivity σ on the temperature and pressure $\sigma \propto T^{3/2}P^{-1/2}$ describes the actual dependence rather well. Therefore, it is assumed below that n = 1 though the case $n \neq 1$ can be investigated in a quite similar way. From n = 1 it follows that

$$u^2 = -(2-\gamma)\gamma^{-1} aN \ln(1-F^2) - ab(1-F^2)^{-1} + D$$

the constant D being determined from the condition $u(F_0) = 0$, which follows from (19).

As a result we obtained

$$\left(\frac{dF}{d\varphi}\right)^2 = \Phi(F)$$

$$\Phi(F) = (1-F^2)^2\left\{\frac{2-\gamma}{\gamma} aN \ln(1-F_0^2) + ab(1-F_0^2)^{-1} - \frac{2-\gamma}{\gamma} aN \ln(1-F^2) - ab(1-F^2)^{-1}\right\} \tag{21}$$

$$\Phi(F_0) = 0$$

For the periodicity of the function $F(\varphi)$ and a variation between the limits $F_0 \leq F \leq F_*$, ($F_* < 1$) it is necessary that the curve $\Phi(F)$ have the form shown schematically in Fig. 1 (solid portion), that is, it must satisfy the following conditions:

$$\Phi(F_0) = 0, \; \Phi(F_*) = 0, \; \Phi(F) > 0 \qquad (F_0 < F < F_*) \tag{22}$$

The first of conditions (22) is satisfied automatically; for the third condition to be fulfilled it is necessary that $\Phi'(F_0) > 0$. Calculation of the derivative gives

$$\Phi'(F) = 2F(1-F^2)\left\{\frac{2-\gamma}{\gamma}aN + ab\left(\frac{1}{1-F^2} - \frac{2}{1-F_0^2}\right) + 2\frac{2-\gamma}{\gamma}aN \ln\frac{1-F^2}{1-F_0^2}\right\}$$

$$\Phi'(F_0) = 2F_0(1-F_0^2)\left[\frac{2-\gamma}{\gamma}aN - ab(1-F_0^2)^{-1}\right]$$

From this it is evident that if the conditions

$$(2-\gamma)\gamma^{-1}aN > ab, \quad 0 < F_0 < F_{0*}, \quad F_{0*}^2 = 1 - \gamma/(2-\gamma)\, ab/aN \tag{23}$$

are fulfilled, then the condition $\Phi'(F_0) > 0$ is satisfied. If $\gamma^{-1}(2-\gamma)aN < ab$, then there exists no value $F_0 < 1$, for which $\Phi'(F_0) > 0$; therefore, a solution for which $(2-\gamma)\gamma^{-1}aN < ab$ is without physical meaning. At first glance the second root F_* of the equation $\Phi(F) = 0$ seems to coincide with the point $F = 1$, as $\Phi(1) = 0$. However, examination of the derivative shows that

$$\lim_{F\to 1} \Phi'(F) = ab > 0$$

so that as $F \to 1$, the function $\Phi(F) \to 0$ through negative values, as indicated by the dashed curve in Fig. 1. Therefore, when conditions (23) hold, a root F_*, for which $F_* = F_*(F_0) < 1$, necessarily exists. This implies that when conditions (23) hold there exist periodic solutions that vary from F_0 at $\varphi = 0$ to $F_*(F_0)$. The value F_* is attained for some value of the argument $\varphi = \varphi_0$, which represents a half-period that must be equated to π/i, i.e.,

$$\int_{F_0}^{F_*(F_0)} \frac{dF}{\sqrt{\Phi(F)}} = \frac{\pi}{i} \qquad (i = 1, 2, 3, \ldots)$$

This condition can serve for the choice of the arbitrary constant F_0. In practice it is inconvenient to use this relationship; it is better to determine F_0 by integrating Eq. (15) numerically with the initial conditions (19) and varying F_0 between the limits (23) until coincidence of the argument φ_0 with one of the values π/i is obtained.

To illustrate what has been said above, a numerical integration of Eq. (15) was performed for the value $(2-\gamma)\gamma^{-1}aN = 50$, which satisfies the first of conditions (23). The dependence of the half-period φ_0 on the initial value F_0 that was obtained is shown in Fig. 2. The intersection points of the curve $\varphi_0(F_0)$ with the lines $\varphi_0 = \pi/i$ determine values of F_{0i} that ensure the required periodicity of the function $F(\varphi)$. On the plot these points are denoted by small circles.

The functions $F(\varphi)$, corresponding to the values F_{0i}, are represented in Fig. 3. The curves are drawn for the half-period $0 \leq \varphi \leq \pi/i$, as the second half of a curve is symmetric with respect to the point $\varphi = \pi/i$. According to relation (18), which in the present case $(m = -3/2, n = 1)$ results in the form $\xi_*^5 F(\varphi) = \text{const}$, the function $F(\varphi)$ determines the shape of the transverse cross section of the cloud. The patterns obtained for the functions $F(\varphi)$ shown in Fig. 3 are represented in Fig. 4. From these it is evident that the greater the number of lobes, the more closely the shape of the transverse cross section approximates a circle.

In conclusion the author thanks V. I. Khonichev for assistance with the numerical calculations.

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